Symplectic geometry of Riemann-Hilbert correspondences

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Outline

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Classically the RH-correspondence relates algebraic vector bundles of rank r on a complex smooth projective curve X endowed with algebraic connections with RS at a finitely many points $S := \{x_1, ..., x_n\}$ (de Rham side) and representations of the fundamental group $\pi_1(X - S, x), x \in X - S$ on $GL(r, \mathbb{C})$ (Betti side).
Chain of generalizations, most notably by Deligne and Kashiwara (with collaborators), has lead to the proof of RH-correspondence for holonomic D-modules in higher dimensions, possibly with irregular singularities. Roughly, conventional RH is a statement about the (derived) equivalence of the category of holonomic D-modules and the category of constructible sheaves.

The equivalence functor (RH functor) assigns to a system of differential equations on a smooth complex algebraic variety X (e.g. to a single equation $P(x, D_x)u = 0$) its sheaf of solutions. At the derived level it maps the corresponding D-module $\mathcal{M}$ to $\mathbf{R}Hom_{D_X}(\mathcal{M}, \mathcal{O}_X)$. The singular support (characteristic variety) $SS(\mathcal{M})$ is a conic coisotropic subvariety of T^*X. For holonomic D-modules the variety $SS(\mathcal{M})$ is Lagrangian. Hence it is an object of an appropriately defined Fukaya category of T^*X. In the irregular case the corresponding object should depend on the "Stokes data". Similarly, one can work with $\hbar$-D-modules which can be thought of as a deformation quantization (defined over $\mathbb{C}[\hbar]$) of the category of coherent sheaves on T^*X. Then we can have not necessarily conic Lagrangians $Supp(\mathcal{M}) \subset T^*X$ obtained as the support of the reduction modulo $\hbar$. In the language of PDEs it can be seen via the WKB asymptotics as $\hbar \to 0$.

Log-extension of the cotangent bundle of a curve

In general the relation between Fukaya categories and quantized complex algebraic symplectic manifolds is based on the notion of log-extension of the latter. Roughly, any such extension gives rise to the corresponding category of D-modules, related by RH to an appropriate Fukaya category. I am going to illustrate all that in the case of holonomic D-modules on curves.

Let X be a complex projective curve, $S = \{x_1, ..., x_n\} \subset X$ a finite subset. A singular term at a point $x_i \in S$ is a Puiseux polynomial in negative powers with respect to a local parameter: $c_\alpha(x) = \sum_{\lambda \in \mathbf{Q}_{\leq 0}} c_{\alpha,\lambda}(x - x_i)^\lambda$. Each polynomial c_α has a multiplicity $m_\alpha \geq 1$. Then for a given choice of singular terms with given multiplicities (+some numerical conditions) there is a canonical choice to include the symplectic manifold $M = T^*X$ as an open symplectic leaf to a Poisson manifold M_{log} (log-extension of M).

Construction of the log-extension

First consider the fiberwise projectivization $\overline{T^*X}$. The standard holomorphic symplectic form $\omega^{2,0} = \omega^{2,0}_{T^*X}$ has poles of order 2 at the divisor at infinity $X_\infty \simeq X$. Second, we construct a sequence of blow-ups $W_i = Bl_{p_i}(W_{i-1})$, $W_0 = \overline{T^*X}$ such that p_i is either a smooth point of a divisor in W_{i-1} at which the pull-back of $\omega^{2,0}$ has pole of order ≥ 2, or p_i is the intersection of two divisors where the pull-back of $\omega^{2,0}$ has pole of order ≥ 1. We start the above chain of blow-ups at points of $S_\infty \subset X_\infty$ which is a copy of the set S. Finally we keep only those divisors for which the pull-back of $\omega^{2,0}$ has poles of order 1. The union of T^*X and these divisors is our M_{log}. Details see in our paper arXiv:1303.3253.

Holonomic D-modules with irregular singularities on curves

Figure: blow-ups

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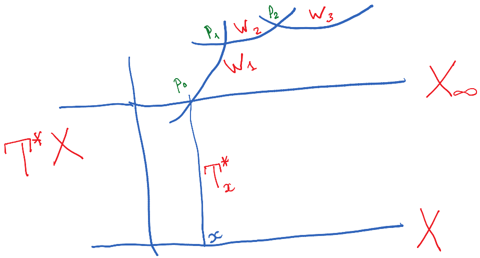

Fix now

Data

- A finite set $S = \{x_i \in X, 1 \leq i \leq n\}$.
- A collections of singular terms $c_\alpha, \alpha \in J$ at these points, each term with multiplicity m_α.
- A complex singular Lagrangian subvariety (loosely called spectral curve) $L \subset M = T^*X$ with compact closure in M_{log}.

Then one can define a $\mathbb{C}[\hbar]$-linear category $Hol((c_\alpha)_{\alpha\in J}, L)$ of holonomic $\hbar$-D-modules with given singular terms $(c_\alpha)_{\alpha\in J}$ at S and given spectral curve L.
Intuitively, (some) objects of the category $Hol((c_\alpha)_{\alpha\in J}, L)$ are $\hbar$-connections on the curve X which modulo $\hbar$ are Higgs bundles with prescribed singular behavior at S and prescribed spectral curve L.
Similarly we define the category $Hol((c_\alpha)_{\alpha\in J})$ of $\hbar$-D-modules for which L is not fixed. For $\hbar \neq 0$ the it is equivalent to the category of holonomic D-modules on X with prescribed singular terms $c_\alpha/\hbar$ at S.
These categories appear on the de Rham side of our RH-correspondences.

Families of Fukaya categories over $\hbar \in \mathbb{C}^*$

On the Betti side we suggest to put (families in $\hbar$ of) categories of finite-dimensional representations of certain (partially wrapped) Fukaya categories.

Let $\omega^{2,0}$ be the standard holomorphic symplectic form on $M = T^*X$. Rescaling $\omega^{2,0} \mapsto \omega^{2,0}/\hbar$ we obtain a family of complex symplectic manifolds over $\hbar \in \mathbb{C}^*$.

There are two kinds of (families over $\hbar \in \mathbb{C}^*$) Fukaya categories in the story: the local one $\mathcal{W}F_{loc,\hbar}(L)$ (symplectic form is $\omega_\hbar = Re(\omega^{2,0}/\hbar)$, B-field $B_\hbar = Im(\omega^{2,0}/\hbar)$) and the global one, $\mathcal{W}F_{glob,\hbar}(M)$. The former is associated with a small tubular neighborhood of a given spectral curve L, while the latter is associated with M considered as an open symplectic leaf of M_{log}.

Both local and global Fukaya categories are the categories of finite-dimensional modules (i.e. A_∞-functors to $D^b(Vect_{\mathbb{C}})$) of the partially wrapped Fukaya categories of these real symplectic manifolds. Objects of $\mathcal{WF}_{glob,\hbar}(M)$ are Lagrangian submanifolds of M with compact closure in M_{log}, endowed with finite-dimensional local systems. Definition of the A_∞-structure is more subtle, since it involves Hamiltonian flows which act on the open part of the log divisor $D_i^\circ \simeq \mathbb{C} = \mathbb{R}^2 \subset M_{log} - M$ as a translation along a real straight line. There are still some foundational questions concerning partially wrapped Fukaya categories, which we will ignore here. In the end the category $\mathcal{WF}_{glob,\hbar}(M)$ is a generalization of the Nadler-Zaslow Fukaya category of T^*X.

Global RH-correspondence

There are two types of RH-correspondences in our story: local (when the spectral curve L is fixed) and global.
Global RH-correspondence is essentially a reformulation of the RH-correspondence for holonomic irregular D-modules on curves formulated by Deligne and Malgrange (80's). I formulate the result as a conjecture (some details of the proof have to be verified).

Conjecture

The categories $Hol((c_\alpha))$ and $\mathcal{W}F_{glob,\hbar}(M)$ are equivalent (as triangulated A_∞-categories over the Novikov field $\mathbb{C}((e^{\mathbb{R}/\hbar})) = \sum_{\lambda_i \to +\infty} c_i e^{-\lambda_i/\hbar}, c_i \in \mathbb{C}$, probably even as categories over the field of germs of meromorphic functions at $\hbar = 0$).

Comments on the Conjecture

To prove the conjecture one identifies both sides with categories of constructible sheaves. For each $\hbar \in \mathbb{C}^*$ on the de Rham side we have the category of holonomic D-modules with singular terms $c_\alpha/\hbar$. Reformulation of Deligne-Malgrange RH-correspondence gives on the Betti side the category of constructible sheaves F such that the microsupport $\mu(F)$ belongs to the union of X and all positive conormal bundles $T^{*,+}_{S^1_\alpha} \subset M$, where S^1_α is a closed cooriented curve $\theta \mapsto e^{Re(c_\alpha/\hbar)} \cdot e^{i\theta}$ understood as the image of a very small circle about a singular point (next slide).

Our Fukaya categories are invariants of the corresponding Legendrian links at infinity. Comparison with the work of Shende-Treumann-Zaslow on Fukaya categories of Legendrian knots goes via a contact isotopy of the neighborhhod of infinity of the (co)spherical cotangent bundle S^*X and the one of the neighborhood of $M_{log} - M$.

Conormal bundles

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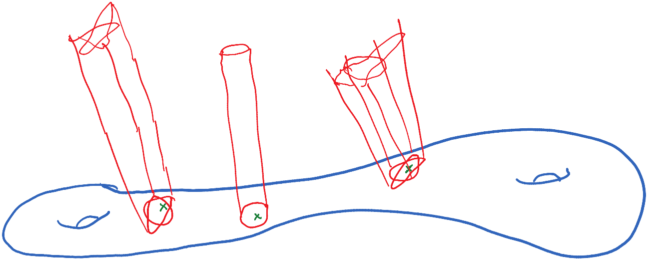

Local RH

In the local RH-correspondence we fix L. The corresponding Fukaya category is defined over $\mathbb{Z}$ since there are no pseudo-holomorphic discs with the boundary on L in a small neighborhood of L.

Conjecture

Assume that $\hbar$ does not belong to a Stokes ray $l_\gamma = \mathbb{R}_{>0} exp(\int_{\gamma \in H_2(M,L,\mathbb{Z})} \omega^{2,0})$. Then there is an embedding $\mathcal{W}F_{loc,\hbar}(L) \to \mathcal{W}F_{glob,\hbar}(M)$.
Moreover, $Hol((c_\alpha)_{\alpha \in J}, L)$ is equivalent to $\mathcal{W}F_{loc,\hbar}(L)$ as categories over $\mathbb{C}((\hbar))$ (or even meromorphically). This local equivalence is compatible with the global one outside of Stokes rays.

Inverse RH-functor is given by "family Floer homology"-type construction.

Holonomic modules over quantum tori

Consider the quantum torus, which is the algebra $A_q(n)$, $0 < |q| < 1$ with invertible generators $x_i, y_i, 1 \leq i \leq n$ and relations $x_i y_j = q^{\delta_{ij}} y_j x_i, 1 \leq i, j \leq n$. It is useful to think that $q = e^{\hbar}$. The theory of D_q-modules makes sense on the torus $(\mathbb{C}^*)^n$. Analog of the cotangent bundle for D_q-modules is $(C^*)^{2n}$. In many respects the theory of D_q-modules is similar to the theory of algebraic D-modules on $\mathbb{C}^n$ (e.g. there exists an analog of Bernstein filtration, notion of holonomic D_q-modules, direct and inverse image functors, b-function, etc.). It was developed in 90's by Sabbah.
The RH-correspondence is less understood. In the case $n = 1$ the analog of the Stokes data was proposed relatively recently by Ramis, Saloy and Zhang (arXiv:0903.0853).

Let $\mathcal{M}$ is a D_q-module on $\mathbb{C}^*$ (physicists know such objects under the name of quantum spectral curves). The corresponding sheaf of solutions V is defined similarly to the case of usual D-modules. It is a coherent sheaf on the elliptic curve $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$. Let *qHol* denotes the category of holonomic D_q-modules on $\mathbb{C}^*$.
Let $0 = V_{\leq\lambda_m} \subset V_{\leq\lambda_{m-1}} \subset ... \subset V_{\leq\lambda_0=\infty} := V, \lambda_i \in \mathbb{R} \cup \{\infty\}$ be a finite filtration of V by coherent subsheaves such that all quotients $F_{\leq\lambda_i} := V_{\leq\lambda_i}/V_{\leq\lambda_{i+1}}$ are semistable and their slopes strictly increase (e.g. V_∞/V_{λ_1} is a torsion sheaf). We call such a filtration anti-HN-filtration (since the slopes satisfy opposite to Harder-Narasimhan inequalities).

Theorem

The category of holonomic D_q-modules on $\mathbb{C}^*$ is equivalent to the category of coherent sheaves on the elliptic curve E_q, which are endowed with two anti-HN filtrations labeled by $\mathbf{Q} \cup \{\infty\}$.

These anti-HN filtrations are analogs of the Stokes data at $x = 0$ and $x = \infty$. The role of singular terms c_α is now played by rational numbers. This can be also seen from the formal classification of holonomic D_q-modules over $\mathbb{C}((x))$.

Comments on the Theorem

How to describe the D_q-module corresponding to the given coherent sheaf endowed with two aHN filtrations?
Suppose we are given a coherent sheaf F on $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ endowed with two filtrations $(F^0_{\leq\lambda})$ and $(F^{\infty}_{\leq\lambda})$, where $\lambda \in \mathbb{R} \cup \{\infty\}$ such that
$0 \subset F^0_{\leq\lambda^0_{k_m}} \subset F^0_{\leq\lambda^0_{k_{m-1}}} \subset ... \subset F^0_{\leq\lambda^0_1} \subset F^0_{\leq\infty} = F$ (and similarly with the superscript 0 being replaced by ∞), such that the consequitive terms of the filtrations are different, the "slopes" are rational numbers which are ordered such as follows: $\lambda^0_1 > ... > \lambda^0_{k_m}$ (and similarly for the filtration at ∞), the quotient $F/F^0_{\leq\lambda^0_1}$ is a torsion sheaf on E_q (and similarly for the filtration at infinity) and all other consequitive quotients are semistable vector bundles of the slopes λ^0_i (and similarly for λ^{∞}_i).

Then the corresponding D_q-module $\mathcal{M}$ is described as a vector space of analytic sections of the coherent sheaf which is the pull-back of F to $\mathbb{C}^*$, endowed with the natural action of the quantum torus $A_q(1) = \{xy = qyx\}$ (notice: the pull-back of F is a $q^{\mathbb{Z}}$-equivariant sheaf). Furthermore, the restrictions of these sections to the pull-backs to $\mathbb{C}^*$ of the associated graded quotients of the filtrations of F satisfy the following properties:
a) they are trivial on the first terms (i.e. torsion) near $x = 0$ for $(F^0_{\leq\lambda})$ and near $x = \infty$ for $F^{\infty}_{\leq\lambda}$;
b) for any $i \geq 1$ the induced sections of the pull-backs of $F^0_{\leq\lambda^0_i}/F^0_{<\lambda^0_i}$ grow (as $|x| \to 0$) as $c^0_1|x|^{c^0_2} q^{\frac{1}{2}\left(\frac{log|x|}{log|\lambda^0_i|}\right)^2}$ for some $c^0_1, c^0_2 > 0$, and similar asymptotics as $|x| \to \infty$ with λ^0_i being replaced by λ^{∞}_i.

In the higher-dimensional case we propose a conjectural answer for the Betti side of RH. It is a mixture of tropical, symplectic and algebraic geometry.

Let $L \subset \mathbb{R}^{2n}$ be a closed conic polyhedral Lagrangian subset, i.e. a finite union of closed rational convex Lagrangian cones $L_i, i \in I$ of full dimension n. By taking a subdivision we may assume that the boundary of each of each cone does not intersect the interior of another one. Thus we have a "Lagrangian fan".

Let us fix the branch Arg of the standard holomorphic volume form on $\mathbb{R}^{2n} = \mathbb{C}^n$. In order to formulate the qRH-correspondence we will need:

a) The local Fukaya category $\mathcal{F}_L := \mathcal{F}_{L,Arg}$ associated with the above choices.

b) A collection of functors $\Phi_{L_i} := F_{L_1,Arg} : \mathcal{F}_L \to D^b(Vect_{\mathbb{C}})$.

Having the data a), b) for any rational Lagrangian subspace $\alpha \in \mathbb{R}^{2n}$ we will define an abelian subcategory $\mathcal{A}_\alpha := \mathcal{A}_{L,\alpha,Arg} \subset D^b(E_q^n)$ and then propose a conjectural RH-correspondence for holonomic D_q-modules in terms of such abelian categories. More precisely, let us assume for a moment a) and b) as "black boxes" and describe the main conjecture. Notice that the category $D^b(E_q^n)$ is endowed with the natural action of the group $\widetilde{Sp(2n,\mathbb{Z})}$, which the central extension by $\mathbb{Z}$ of the group $Sp(2n,\mathbb{Z})$. The same group acts on the set of pairs (α, Arg). One can show that the action of $\widetilde{Sp(2n,\mathbb{Z})}$ on the set of pairs is transitive. Thus we may assume that modulo this action the subspace α is the standard coordinate subspace α_0 given explicitly by the equations $p_1 = p_2 = ... = p_n = 0$.

The above standard coordinate subspace corresponds to the t-structure in $D^b(E_q^n)$ with the heart consisting of coherent sheaves with finite support. For an arbitrary pair (α, Arg) we define the corresponding t-structure using the transitive action of $\widetilde{Sp(2n,\mathbb{Z})}$. In particular, the category $\mathcal{A}_\alpha$ consists of coherent sheaves which have finite support after identificication (via $g \in \widetilde{Sp(2n,\mathbb{Z})}$) of α with the above standard coordinate subspace.

The Betti side of *qRH*

The idea of *qRH*-correspondence is explained by the observation that since $(\mathbb{C}^*)^{2n} \simeq \mathbb{R}^{2n} \times (S^1)^{2n}$ then the Fukaya category should be a tensor product of the Fukaya category of $\mathbb{R}^{2n}$ and $D^b(E_q^n)$ (via HMS). If we fix a polyhedral Lagrangian $L \subset \mathbb{R}^{2n}$ then we should use the local Fukaya category $\mathcal{F}_L$.
Also, each rational cone L_i which appears in the definition of L gives rise to the t-structure in $D^b(E_q^n)$ via the Lagrangian subspace $\alpha_i \supset L_i$.
Let us fix a rational Lagrangian vector space $\alpha \in \mathbb{R}^{2n}$. Let $\mathcal{B}_L \subset \mathcal{F}_L \otimes D^b(E_q^n)$ be the full subcategory consisting of objects E such that $(\Phi_{L_i} \otimes id)(E)$ belongs to the category $\mathcal{A}_{\alpha_i}$ for each $L_i, i \in I$. We define $\mathcal{B}$ as the inductive limit of $\mathcal{B}_L$ with respect to the natural embeddings of Lagrangian subvarieties.

qRH-correspondence

Conjecture

The category of holonomic D_q-modules on $(\mathbb{C}^*)^n$ is equivalent to $\mathcal{B}$.

About the Fukaya category

In order to complete the story we are going to describe the data a) and b).
Let us start with the Fukaya category $\mathcal{F}_L$. Notice that because L is conic it defines a Legendrian subvariety $\Lambda := \Lambda_L \subset S^{2n-1}$, where we understand the "sphere at infinity" S^{2n-1} as a contact manifold, endowed with the induced from $\mathbb{R}^{2n}$ contact structure.
Let us choose a generic (i.e. irrational) Lagrangian vector subspace β in $\mathbb{R}^{2n}$. Its intersection with S^{2n-1} gives rise to a Legendrian sphere S_β^{n-1}. Because of genericity of β it does not intersect Λ (but they can have non-trivial linking, even in the case $n = 1$).

Proposition

The manifold $S^{2n-1} - S_\beta^{n-1}$ is contactly isomorphic to the spherical bundle $S^*(B^n) = T^*B^n/\mathbb{R}_{>0}$ over the standard ball $B^n \subset \mathbb{R}^n$.

We *define* $\mathcal{F}_L$ via the above Proposition using the work of Shende-Treumann-Zaslow. In that case, the category $\mathcal{F}_L$ can be described in terms of constructible sheaves.
Explicitly, it can be spelled out such as follows. Let us identify Λ with its image under the contact isotopy from the Proposition. Let $pr : S^*(B^n) \to B^n$ be the natural projection. Then $pr(\Lambda) \cap B^n$ is a cooriented compact hypersurface with the boundary. The cone decomposition $L = \cup_i L_i$ gives rise to a real-analytic constructible stratification $(N_i)_{i \in I}$ of B^n, induced by the projection pr. Then our category is equivalent to the full subcategory Sh_L of such $E \in D^b_{constr}(B^n)$ that the microsupport $\mu(E) \subset \cup_i T^{*,+} N_i$, and which are quasi-isomorphic to the zero sheaf in a neighborhood of ∂B^n. Here $T^{*,+} N_i$ denotes the positive (with respect to the coorientation) cotangent bundle.

Example: $n = 1$

For $n = 1$ the ball B^1 is just an closed interval $[a, b] \subset \mathbb{R}$. The strata N_i are points $x_i \in [a, b]$, endowed with the signs $\pm$ which reflect the coorientation. Then the Fukaya category in question is equivalent to the subcategry of $D^b_{constr}([a, b])$ consisting of complexes which have trivial cohomology in some neighborhoods of the boundary points a, b and whose microsupport belongs to the union of vertical rays outcoming from the points x_i. The ray goes up if the point x_i is assigned $+$ and it goes down, if it is assigned $-$.

Interval and rays

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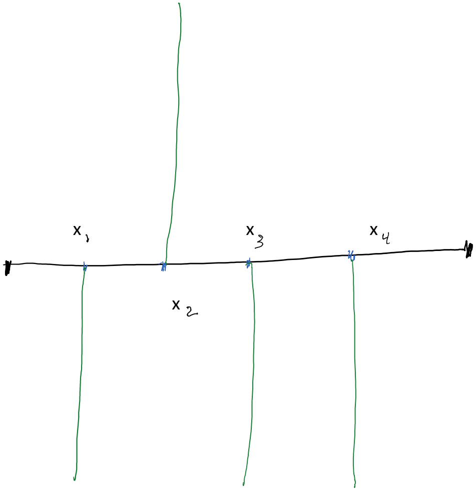

Functors

Finally, we describe the functors Φ_{L_i} from the condition b) by taking the fiber of the corresponding constructible sheaf at any smooth point of the stratum N_i (the answer does not depend on the point).

Remark

We can avoid using the Hamiltonian isotopy with the cospherical bundle by utilizing $\mathcal{W}F_{loc}(L)$, as we did with D-modules. This gives in the end the A-model description of the category $\mathcal{B}$.

Family Floer homology and the *qRH*-functor

The construction of the coherent sheaf from a D_q-module can be spelled out symplectically.
Namely, consider the family of Floer homology $Hom(L, (\mathbb{C}^*_x, E))$, where $\mathbb{C}^*_x, x \in \mathbb{C}^*$ is the transversal to L family of Lagrangians, and E is a rank 1 local system. This gives a 2-parameter family of vector spaces. The family is constant in log-coordinates along the 1-dimensional foliation induced by linear shifts along the directions x and E. The quotient of $(\mathbb{C}^*)^2$ by this foliation is our elliptic curve. Hence the Floer family homology construction gives a coherent sheaf on the elliptic curve. If it is a vector bundle, the Ramis-Saloy-Zhang result endows it with two anti-HN-filtrations.